Problem 1.29

Do the same tasks as in Problem 1.28 but for the case of four particles ($N = 4$).

Solution

Suppose there are N particles in a system. One of the particles, labelled α, experiences a force from the remaining $N - 1$ particles in addition to an external force.

$$\text{Net Force on Particle } \alpha = \mathbf{F}_\alpha = \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}}$$

If $N = 4$, then

$$\mathbf{F}_\alpha = \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{4} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} \quad \Rightarrow \quad \begin{cases} \mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}} \\[2mm] \mathbf{F}_2 = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}} \\[2mm] \mathbf{F}_3 = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}} \\[2mm] \mathbf{F}_4 = \mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}} \end{cases}.$$

According to Newton's second law, the net force on a particle is equal to the rate of change of momentum.

$$\begin{cases} \dfrac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}} \\[3mm] \dfrac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}} \\[3mm] \dfrac{d\mathbf{p}_3}{dt} = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}} \\[3mm] \dfrac{d\mathbf{p}_4}{dt} = \mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}} \end{cases}$$

Add them together to get the rate of change of total momentum.

$$\frac{d\mathbf{P}}{dt} = \sum_{\beta=1}^{4} \frac{d\mathbf{p}_\beta}{dt}$$

$$= \left(\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}}\right) + \left(\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}}\right) + \left(\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}}\right)$$

$$+ \left(\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}}\right)$$

$$= \underbrace{\left(\mathbf{F}_{12} + \mathbf{F}_{21}\right)}_{=\,0} + \underbrace{\left(\mathbf{F}_{13} + \mathbf{F}_{31}\right)}_{=\,0} + \underbrace{\left(\mathbf{F}_{14} + \mathbf{F}_{41}\right)}_{=\,0} + \underbrace{\left(\mathbf{F}_{23} + \mathbf{F}_{32}\right)}_{=\,0}$$

$$+ \underbrace{\left(\mathbf{F}_{24} + \mathbf{F}_{42}\right)}_{=\,0} + \underbrace{\left(\mathbf{F}_{34} + \mathbf{F}_{43}\right)}_{=\,0} + \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}}$$

By Newton's third law, $\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0}$.

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}}$$

Therefore, if the net external force acting on the particles is zero,

$$\frac{d\mathbf{P}}{dt} = \mathbf{0},$$

then the total momentum is conserved: $\mathbf{P}_{\text{initial}} = \mathbf{P}_{\text{final}} = \text{constant}$.